Area by Double Integration

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

Overview

We discuss in the lecture how to use double integrals to calculate the areas of bounded regions in the plane and to find the average value of a function of two variables.

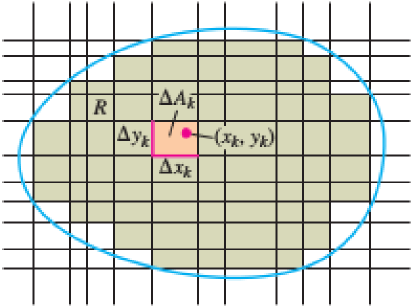
Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k.$$

This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R.

As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete.



Areas of Bounded Regions in the Plane

We define the area of R to be the limit

$$\text{Area} = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA.$$

Definition 1.

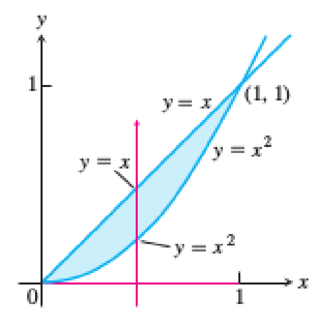
The area of a closed, bounded plane region R is

$$A = \iint_R dA.$$

To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R.

Exercise 2.

1. *Find the area of the region R bounded by* $y = x^2$ *and* $y = x$ *in the first quadrant.*

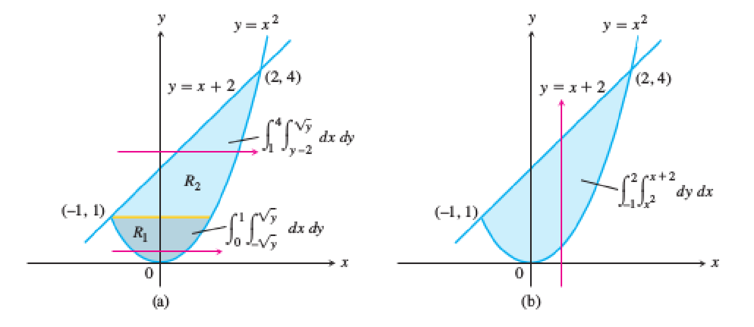


Solution

1. $A = \int_0^1 \int_{x^2}^{x} dy \; dx = \dfrac{1}{6}.$

Exercise 3.

2. *Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.*



Note that the second integral requires only one integral.

Solution

2. $A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \, dx \ dy + \int_1^4 \int_{y-2}^{\sqrt{y}} \, dx \ dy = \int_{-1}^2 \int_{x^2}^{x+2} \, dy \ dx$

Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region in the plane, the average value is the integral over the region divided by the area of the region.

This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of R.

Average Value

We are led to define the average value of an integrable function f over a region R to be

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint\limits_R \; dA.$$

If f is the temperature of a thin plate covering R, then the double integral of f over R divided by the area of R is the plate's average temperature.

If $f(x, y)$ is the distance from the point (x, y) to a fixed point P, then the average value of f over R is the average distance of points in R from P.

Exercise 4.

3. *Find the average value of* $f(x, y) = x \cos xy$ *over the rectangle* $R : 0 \leq x \leq \pi,\ 0 \leq y \leq 1.$

Solution

3. The value of the integral of f over R is

$$\int_0^{\pi} \int_0^1 x \cos xy \; dy \; dx = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

Exercise 5.

In the following exercises, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

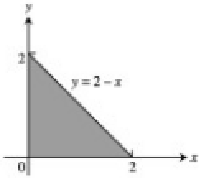
1. *The coordinate axes and the line* $x + y = 2$
2. *The parabola* $x = y - y^2$ *and the line* $y = -x$
3. *The parabolas* $x = y^2$ *and* $x = 2y - y^2$
4. *The lines* $y = 1 - x$ *and* $y = 2$ *and the curve* $y = e^x$
5. *The lines* $y = x - 2$ *and* $y = -x$ *and the curve* $y = \sqrt{x}$

Solution for (1.) in Exercise 5

$$\int_0^2 \int_0^2 dy\ dx = \int_0^2 (2-x)dx$$
$$= [2x - \frac{x^2}{2}]_0^2 = 2$$

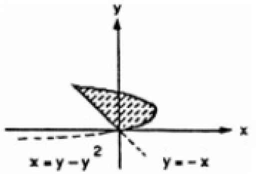
OR

$$\int_0^2 \int_0^{2-y} dx\ dy = \int_0^2 (2-y)dy$$
$$= 2$$



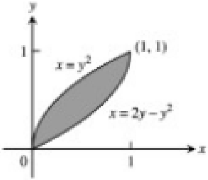
Solution for (2.) in Exercise 5

$$\begin{aligned}
\int_0^2 \int_{-y}^{y-y^2} dx\ dy &= \int_0^2 (2y - y^2) dy \\
&= [y^2 - \frac{y^3}{3}]_0^2 \\
&= 4 - \frac{8}{3} \\
&= \frac{4}{3}
\end{aligned}$$



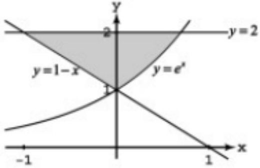
Solution for (3.) in Exercise 5

$$\int_0^1 \int_{y^2}^{2y-y^2} dx\ dy = \int_0^1 (2y - 2y^2) dy$$
$$= [y^2 - \frac{2}{3} y^3]_0^1$$
$$= \frac{1}{3}$$



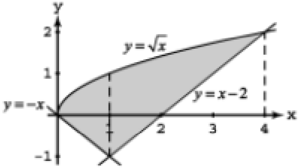
Solution for (4.) in Exercise 5

$$
\begin{aligned}
\int_1^2 \int_{1-y}^{\ln y} 1 \; dx \; dy &= \int_0^2 [x]_{1-y}^{\ln y} dy \\
&= \int_1^2 (\ln y - 1 + y) dy \\
&= \left[y \ln y - 2y + \frac{y^2}{2} \right]_1^2 \\
&= 2 \quad \ln 2 - \frac{1}{2}
\end{aligned}
$$



Solution for (5.) in Exercise 5

$$\int_0^1 \int_{-1}^{\sqrt{x}} 1 dy\ dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\ dx = \int_0^1 [y]_{-x}^{\sqrt{x}} dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} dx$$
$$= \int_0^1 (\sqrt{x} + x) dx + \int_1^4 (\sqrt{x} - x + 2) dx$$
$$= [\frac{2}{3}x^{3/2} + \frac{1}{2}x^2]_0^1 + [\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x]_1^4$$
$$= \frac{13}{3}$$



Identifying the Region of Integration

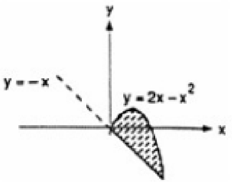
Exercise 6.

The integrals and sums of integrals in Exercises 6-9 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

1. $\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$
2. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$
3. $\int_{-1}^0 \int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-x/2}^{1-x} dy\, dx$
4. $\int_0^2 \int_{x^2-4}^{0} dy\, dx + \int_0^4 \int_0^{\sqrt{x}} dy\, dx$

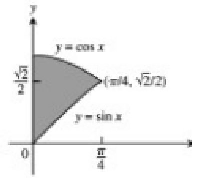
Solution for (1.) in Exercise 6

$$
\begin{aligned}
\int_0^3 \int_{-x}^{2x-x^2} dy\ dx &= \int_0^3 (3x - x^2) dx \\
&= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 \\
&= \frac{27}{2} - 9 \\
&= \frac{9}{2}
\end{aligned}
$$



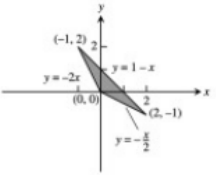
Solution for (2.) in Exercise 6

$$
\begin{aligned}
\int_0^{x/4}\int_{\sin\ x}^{\cos\ x} dy\ dx &= \int_0^{x/4} (\cos\ x - \sin\ x)dx = [\sin\ x + \cos\ x]_0^{x/4} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) \\
&= \sqrt{2} - 1
\end{aligned}
$$



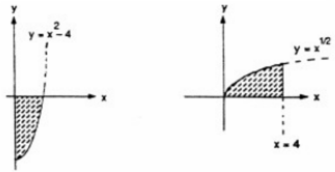
Solution for (3.) in Exercise 6

$$
\begin{aligned}
\int_{-1}^{0}\int_{-2x}^{1-x} dy\ dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\ dx &= \int_{-1}^{0}(1+x)dx + \int_{0}^{2}(1-\frac{x}{2})dx \\
&= [x+\frac{x^2}{2}]_{-1}^{0} + [x-\frac{x^2}{4}]_{0}^{2} \\
&= -(-1+\frac{1}{2}) + (2-1) \\
&= \frac{3}{2}
\end{aligned}
$$



Solution for (4.) in Exercise 6

$$
\begin{aligned}
\int_0^2 \int_{x^2-4}^0 dy\ dx + \int_0^4 \int_0^{\sqrt{4}} dy\ dx &= \int_0^2 (4-x^2)dx + \int_0^4 x^{1/2}dx \\
&= [4x - \frac{x^3}{3}]_0^2 + [\frac{2}{3}x^{3/2}]_0^4 \\
&= (8-\frac{8}{3}) + \frac{16}{3} \\
&= \frac{32}{3}
\end{aligned}
$$



Finding Average Values

Exercise 7.

1. *Find the average value of* $f(x,y) = \sin(x+y)$ *over*
   (a) *the rectangle* $0 \le x \le \pi, \quad 0 \le y \le \pi.$
   (b) *the rectangle* $0 \le x \le \pi, \quad 0 \le y \le \pi/2.$
2. *Find the average height of the paraboloid* $z = x^2 + y^2$ *over the square* $0 \le x \le 2,\ 0 \le y \le 2.$
3. *Find the average value of* $f(x,y) = 1/(xy)$ *over the square* $\ln 2 \le x \le 2\ln 2,\ \ln 2 \le y \le 2\ln 2.$

Solution for the Exercise 7

1. (a) average$= \frac{1}{\pi^2}\int_0^x\int_0^x \sin(x+y)dy\ dx = \frac{1}{\pi^2}\int_0^x[-\cos(x+y)_0^x dx =$ $\frac{1}{\pi^2}\int_0^x[-\cos(x+\pi)+\cos\ x]dx = \frac{1}{\pi^2}[-\sin(x+\pi)+\sin\ x]_0^x =$ $\frac{1}{\pi^2}[(-\sin\ 2\pi+\sin\ \pi)-(-\sin\ \pi+\sin 0)] = 0$

   (b) average$= \frac{1}{(\frac{x^2}{2})}\int_0^x\int_0^{x/2}\sin(x+y)dy\ dx = \frac{2}{\pi^2}\int_0^x[-\cos(x+y)]_0^{\pi/2}dx =$ $\frac{2}{\pi^2}\int_0^x[-\cos(x+\frac{\pi}{2})+\cos x]dx = \frac{2}{\pi^2}[-\sin(x+\frac{\pi}{2})+\sin\ x]_0^x =$ $\frac{2}{\pi^2}[(-\sin\frac{3\pi}{2}+\sin\pi)-(-\sin\frac{\pi}{2}+\sin\ 0)] = \frac{4}{\pi^2}$

2. average height$= \frac{1}{4}\int_0^2\int_0^2(x^2+y^2)dy\ dx = \frac{1}{4}\int_0^2\left[x^2y+\frac{y^3}{3}\right]_0^2dx = \frac{1}{4}\int_0^2(2x^2+\frac{8}{3})dx =$ $\frac{1}{2}\left[\frac{x^3}{3}+\frac{4x}{3}\right]_0^2 = \frac{8}{3}$

3. average$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ \ln 2}\int_{\ln 2}^{2\ln 2}\frac{1}{xy}dy\ dx = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ \ln 2}\left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ \ln 2}dx = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ \ln 2}\frac{1}{x}(\ln 2+$ $\ln\ln 2-\ln\ln 2)dx = \left(\frac{1}{\ln 2}\right)\int_{\ln 2}^{2\ \ln 2}\frac{dx}{x} = \left(\frac{1}{\ln 2}\right)[\ln x]_{\ln 2}^{2\ln 2} = \left(\frac{1}{\ln 2}\right)(\ln 2+\ln\ln 2-\ln\ln 2) = 1$

Bacterium population

Exercise 8.

If $f(x, y) = (10,000e^{y})/(1 + |x|/2)$ *represents the* population density *of a certain bacterium on the xy-plane, where* x *and* y *are measured in centimeters, find the total population of bacteria within the rectangle* $-5 \leq x \leq 5$ *and* $-2 \leq y \leq 0$.

Solution for the Exercise 8

$$
\begin{aligned}
\int_{-5}^{5}\int_{-2}^{0} \frac{10,000e^{y}}{1+\frac{x}{2}}\,dy\ dx &= 10,000(1-e^{-2})\int_{-5}^{5}\frac{dx}{1+\frac{x}{2}} = 10,000(1-e^{-2})\Big[\int_{-5}^{0}\frac{dx}{1-\frac{x}{2}}+\int_{-5}^{0}\frac{dx}{1+\frac{x}{2}}\Big] \\
&= 10,000(1-e^{-2})\Big[-2\ln(1-\frac{x}{2})\Big]_{-5}^{0}+10,000(1-e^{-2})\Big[2\ln(1+\frac{x}{2})\Big]_{0}^{5} \\
&= 10,000(1-e^{-2})\Big[2\ln(1+\frac{5}{2})\Big]+10,000(1-e^{-2})\Big[2\ln(1+\frac{5}{2})\Big] \\
&= 40,000(1-e^{-2})\ln\big(\frac{7}{2}\big) \\
&\approx 43,329
\end{aligned}
$$

Regional population

Exercise 9.

If $f(x, y) = 100(y + 1)$ *represents the population density of a planar region on Earth, where* x *and* y *are measured in miles, find the number of people in the region bounded by the curves* $x = y^2$ *and* $x = 2y - y^2$.

Solution for the Exercise 9

$$\begin{aligned}
\int_0^1 \int_{y^2}^{2y-y^2} 100(y+1)dx\ dy &= \int_0^1 [100(y+1)x]_{y^2}^{2x-y^2} dy \\
&= \int_0^1 100(y+1)(2y-2y^2)dy \\
&= 200 \int_0^1 (y-y^3)dy \\
&= 200\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 \\
&= (200)(\frac{1}{4}) = 50
\end{aligned}$$

Average temperature in Texas

Exercise 10.

According to the Texas Almanac, Texas has 254 counties and a National Weather Service station in each country. Assume that at time t_0*, each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time* t_0*. Your answer should involve information that you would expect to be readily available in the Texas Almanac.*

Solution for the Exercise 10

Let (x, y) be the location of the weather station in country i for $i = 1, 2, \ldots, 254$.

The average temperature in Texas at time t_0 is approximately

$$\frac{\sum_{i=1}^{254} T(x_i, y_i) \triangle_i A}{A}$$

where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county $i, \triangle_i A$ is the area of county i, and A is the area of Texas.

Exercise 11.

If $y = f(x)$ *is a nonnegative continuous function over the closed interval* $a \leq x \leq b$, *show that the double integral definition of area for the closed plane region bounded by the graph of* f, *the vertical lines* $x = a$ *and* $x = b$, *and the* x*-axis agrees with the definition for area beneath the curve.*

Solution for the Exercise 11

Let $y = f(x)$ be a nonnegative, continuous function on $[a, b]$, then

$$A = \iint_R dA = \int_a^b \int_0^{f(x)} dy \ dx = \int_a^b [y]_0^{f(x)} dx = \int_a^b f(x) dx.$$

References

1. M.D. Weir, J. Hass and F.R. Giordano, Thomas' Calculus, 11th Edition, Pearson Publishers.
2. R. Courant and F.John, Introduction to calculus and analysis, Volume II, Springer-Verlag.
3. N. Piskunov, Differential and Integral Calculus, Vol I & II (Translated by George Yankovsky).
4. E. Kreyszig, Advanced Engineering Mathematics, Wiley Publishers.